Chapter 5

Differentiation

We assume the student knows how to take derivatives and is familiar with the notion of a tangent line from calculus. The Mean Value Theorem in Section 5.2 is this chapter's most important theorem. A second important fact (Section 5.1) is that a derivative, whether continuous or not, satisfies the intermediate value property. Taylor's Theorem (Section 5.3) is needed for Taylor series in Chapter 9. For approximations, the remainder term given in Theorem 5.6 is generally easier to use than the remainder term given in Exercise 5.3.8. Since we consider L'Hôpital's rule a tool rather than a major theorem, our proof of L'Hôpital's rule is very general; we do all cases at once. For some it may be more understandable to do each case separately. Example 5.7 shows that L'Hôpital's rule cannot always be applied.

Possible student take home problems or projects are Exercises 5.1.12, 5.2.6, 5.2.9, 5.3.6, 5.4.4, and 5.4.8. See the end of Section 5.2 in this manual for another possibility.

5.1 The Derivative

1. From calculus, $f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$ for $x > 0$. Alternatively, for $c > 0$,

$$\begin{aligned} f'(c) &= \lim_{x\to c}\frac{f(x)-f(c)}{x-c} = \lim_{x\to c}\frac{\sqrt{x}-\sqrt{c}}{x-c} \\ &= \lim_{x\to c}\frac{\sqrt{x}-\sqrt{c}}{x-c}\cdot\frac{\sqrt{x}+\sqrt{c}}{\sqrt{x}+\sqrt{c}} \\ &= \lim_{x\to c}\frac{1}{\sqrt{x}+\sqrt{c}} = \frac{1}{2\sqrt{c}}. \end{aligned}$$

$f'(0) = \lim_{x\to 0} \dfrac{\sqrt{x}-\sqrt{0}}{x-0} = \lim_{x\to 0} \dfrac{1}{\sqrt{x}} = +\infty$. (Note that $x > 0$ here.) As indicated in the Remark following Definition 5.1, the tangent line to f at 0 is vertical.

2. We have $f, g : I \to \mathbb{R}$ are both differentiable at $c \in I$.
<u>To show</u>: $f + g$ is differentiable at c.

$$\begin{aligned}(f+g)'(c) &= \lim_{x\to c} \frac{f(x)+g(x)-(f(c)+g(c))}{x-c} \\ &= \lim_{x\to c} \frac{f(x)-f(c)}{x-c} + \lim_{x\to c} \frac{g(x)-g(c)}{x-c} \\ &= f'(c)+g'(c).\end{aligned}$$

For part 2,

$$\begin{aligned}(fg)'(c) &= \lim_{x\to c} \frac{f(x)g(x)-f(c)g(c)}{x-c} \\ &= \lim_{x\to c} \frac{f(x)g(x)-f(x)g(c)+f(x)g(c)-f(c)g(c)}{x-c} \\ &= \lim_{x\to c} \left[f(x)\frac{g(x)-g(c)}{x-c} + g(c)\frac{f(x)-f(c)}{x-c} \right] \\ &= f(c)g'(c)+g(c)f'(c)\end{aligned}$$

where $\lim_{x\to c} f(x) = f(c)$ since f is continuous at c by Theorem 5.1.
For part 3, let g be the constant function a on I. Since the derivative of a constant function is 0 (this follows directly from Definition 5.1),

$$\begin{aligned}(af)'(c) &= (gf)'(c) = g(c)f'(c)+f(c)g'(c) \\ &= af'(c)+f(c)(0) \\ &= af'(c).\end{aligned}$$

For the rest of part 1, letting $a = -1$ in part 3,

$$(f-g)'(c) = (f+(-g))'(c) = f'(c)+(-g)'(c) = f'(c)-g'(c).$$

3. This follows directly from Proposition 5.1.

4. Using part 2 of Proposition 5.1 to take the derivative of $(fg)(x) = 1$, we obtain $f(x)g'(x) + f'(x)g(x) = 0$. Dividing this by 1, we obtain $0 = \dfrac{f(x)g'(x)}{f(x)g(x)} + \dfrac{f'(x)g(x)}{f(x)g(x)} = \dfrac{g'(x)}{g(x)} + \dfrac{f'(x)}{f(x)}$.

5. By Exercise 4.1.3, f is continuous on $\mathbb{R}$. For $x \neq 0$,

$$f'(x) = x\left(\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) + \sin\frac{1}{x} = \sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x}.$$

$$f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{x\sin\frac{1}{x}}{x} = \lim_{x\to 0}\sin\frac{1}{x},$$

which does not exist by Example 4.13.

6. $f(x) - f(c)$ may be 0 for some x in any neighborhood of c.

7. Suppose f has a local minimum at an interior point $c \in I$, and f is differentiable at c. Then $-f$ has a local maximum at c and $-f$ is differentiable at c. From the proof of proposition 5.2 in the local maximum case, $(-f)'(c) = 0$ and so $f'(c) = 0$. (One could also mimic the proof in Proposition 5.2.)

8. If $f(x) = 0\ \forall\, x \in [a,b]$, then $f'(c) = 0\ \forall\, c \in (a,b)$. So assume f is not identically 0 on $[a,b]$. By Theorem 4.2 f has an absolute maximum and an absolute minimum on $[a,b]$. Since f is not identically zero, at least one of these occurs at an interior point $c \in (a,b)$. By Proposition 5.2, $f'(c) = 0$.

9. Let $c \in I$.

   (a) $x > c \Rightarrow \dfrac{f(x)-f(c)}{x-c} \geq 0$ and $x < c \Rightarrow \dfrac{f(x)-f(c)}{x-c} \geq 0$. Hence, $f'(c) = \lim_{x\to c}\dfrac{f(x)-f(c)}{x-c} \geq 0$.

   (b) f monotone decreasing on $I \Rightarrow -f$ is monotone increasing on I. By (a), $(-f)'(c) \geq 0$ and so $f'(c) \leq 0$. (One could also mimic the argument in (a).)

10. Since f is differentiable on I, Corollary 5.1 implies that all discontinuities of f' in I are of the second kind. Since f' is monotone on I, Corollary 4.4 implies that f' has no discontinuities in I of the second kind. Therefore, f' has no discontinuities in I.

11. No such function exists by Theorem 5.3. For instance, $\dfrac{h}{2}$ would not be in the range of f'.

12. Allowing $\binom{0}{0} = 1$, the formula works for $n = 0$. For $n = 1$ the formula is just the product rule, Proposition 5.1, part 2. Let $k \geq 0$ and assume $(fg)^{(k)} = \sum_{j=0}^{k} \binom{k}{j} f^{(j)} g^{(k-j)}$. (Dropping the x simplifies notation.) <u>To show</u>: $(fg)^{(k+1)} = \sum_{j=0}^{k+1} \binom{k+1}{j} f^{(j)} g^{(k+1-j)}$.

$$
\begin{aligned}
(fg)^{(k+1)} &= \text{first derivative of } (fg)^{(k)} \\
&= \sum_{j=0}^{k} \binom{k}{j} [\text{first derivative of } f^{(j)} g^{(k-j)}] \qquad \text{(induction hypothesis)} \\
&= \sum_{j=0}^{k} \binom{k}{j} f^{(j)} g^{(k-j+1)} + \sum_{j=0}^{k} \binom{k}{j} f^{(j+1)} g^{(k-j)} \qquad \text{(product rule)} \\
(\text{letting } i &= j+1 \text{ in the second part and then replacing } i \text{ by } j) \\
&= \sum_{j=0}^{k} \binom{k}{j} f^{(j)} g^{(k-j+1)} + \sum_{j=1}^{k+1} \binom{k}{j-1} f^{(j)} g^{(k-j+1)} \\
&= \binom{k}{0} f^{(0)} g^{(k+1)} + \sum_{j=1}^{k} \left[\binom{k}{j} + \binom{k}{j-1}\right] f^{(j)} g^{(k-j+1)} \\
&\quad + \binom{k}{k} f^{(k+1)} g^{(0)} \\
&= \binom{k+1}{0} f^{(0)} g^{(k+1)} + \sum_{j=1}^{k} \binom{k+1}{j} f^{(j)} g^{(k-j+1)} \\
&\quad + \binom{k+1}{k+1} f^{(k+1)} g^{(0)} \\
\left(\text{since } \binom{k+1}{j} \right. &= \left. \binom{k}{j} + \binom{k}{j-1} \text{ from combinatorics}\right) \\
&= \sum_{j=0}^{k+1} \binom{k+1}{j} f^{(j)} g^{(k+1-j)}.
\end{aligned}
$$

5.2 Mean Value Theorems

1. (a) By the Mean Value Theorem (MVT), $\exists\, c_1 \in (0, 1)$ such that $1 =$

$f(1) - f(0) = f'(c_1)(1-0) = f'(c_1)$.
(b) By the MVT, $\exists\, c_2 \in (1,2)$ such that $0 = f(2) - f(1) = f'(c_2)(2 - 1) = f'(c_2)$.
(c) By the intermediate value property of f', Theorem 5.3, $\exists\, c_3 \in (c_1, c_2) \subset (0,2)$ such that $f'(c_3) = \dfrac{1}{3}$.

2. Let $g(x) = \dfrac{a_n}{n+1}x^{n+1} + \dfrac{a_{n-1}}{n}x^n + \cdots + \dfrac{a_1}{2}x^2 + a_0 x$ on $[0,1]$. Then $g(0) = 0$, $g(1) = 0$ by assumption, and $g'(x) = f(x)\ \forall\, x \in [0,1]$. By the MVT, $\exists\, c \in (0,1)$ such that $0 = g(1) - g(0) = g'(c)(1-0) = g'(c) = f(c)$. Therefore, c is a root of f in $(0,1)$.

3. Let x_1 and x_2 be two consecutive distinct real roots of f. By the MVT, $\exists\, c \in (x_1, x_2)$ such that $0 = f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$, and so $f'(c) = 0$. Since $\exists\, n-1$ consecutive pairs of the roots of f and f' has a root between each pair, f' has at least $n-1$ distinct roots.
Example. Let $f(x) = x^2 + 1$. Then f has no real roots, but $f'(x) = 2x$ has one real root.

4. First note that the equation of the secant line connecting the points $(a, f(a))$ and $(b, f(b))$ is given by $y - f(b) = \dfrac{f(b) - f(a)}{b-a}(x - b)$. Define $\varphi : [a,b] \to \mathbb{R}$ by $\varphi(x) = f(b) + \dfrac{f(b)-f(a)}{b-a}(x-b) - f(x)$. Since f is continuous on $[a,b]$ and differentiable on (a,b), so is φ. Also, $\varphi(a) = \varphi(b) = 0$. By Rolle's Theorem $\exists\, c \in (a,b)$ such that $0 = \varphi'(c) = \dfrac{f(b)-f(a)}{b-a} - f'(c)$ or equivalently, $f'(c)(b-a) = f(b) - f(a)$.

5. For $n \neq m$ in $\mathbb{N}$, by the MVT, $\exists\, c$ between $\dfrac{1}{n}$ and $\dfrac{1}{m}$ such that $|a_n - a_m| = \left|f\left(\dfrac{1}{n}\right) - f\left(\dfrac{1}{m}\right)\right| = |f'(c)|\left|\dfrac{1}{n} - \dfrac{1}{m}\right| \le \left|\dfrac{1}{n} - \dfrac{1}{m}\right|$. Since $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ converges, it is Cauchy. Hence, $(a_n)_{n\in\mathbb{N}}$ is Cauchy. By Theorem 3.12, $(a_n)_{n\in\mathbb{N}}$ converges.

6. We first show that $g'(x) \ge 0\ \forall\, x > 0$. Let $x > 0$. Then $g'(x) = \dfrac{xf'(x) - f(x)}{x^2} \ge 0 \Leftrightarrow xf'(x) - f(x) \ge 0 \Leftrightarrow f'(x) \ge \dfrac{f(x)}{x}$. By the MVT applied to f on $[0,x]$, $\exists\, c \in (0,x)$ such that $f(x) = f(x) - f(0) = f'(c)(x-0) = xf'(c)$. Since $0 < c < x$ and f' is monotone increasing on $(0,\infty)$, $\dfrac{f(x)}{x} = f'(c) \le f'(x)$. Therefore, $g'(x) \ge 0\ \forall\, x > 0$.

Let $0 < a < b < \infty$. By part 2 of Corollary 5.2, g is monotone increasing on $[a, b]$, and so $g(a) \le g(b)$. Therefore, g is monotone increasing on $(0, \infty)$.

7. Let $\lim_{x \to a} f'(x) = L \in \mathbb{R}$, and let $\varepsilon > 0$. Then $\exists\, \delta > 0$ such that $x \in [a, b]$ with $a < x < a + \delta \Rightarrow |f'(x) - L| < \varepsilon$. Let $x \in [a, b]$ with $a < x < a + \delta$. By the MVT applied to f on $[a, x]$, $\exists\, c_x \in (a, x)$ such that $\dfrac{f(x) - f(a)}{x - a} = f'(c_x)$. Since $a < c_x < x < a + \delta$, $\left|\dfrac{f(x) - f(a)}{x - a} - L\right| = |f'(c_x) - L| < \varepsilon$. Therefore, $\lim_{x \to a} \dfrac{f(x) - f(a)}{x - a} = L$. That is, f is differentiable at a and $f'(a) = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a} = L$. For the endpoint b, assume $\lim_{x \to b} f'(x) = M \in \mathbb{R}$. For $a < x < b$, by the MVT applied to f on $[x, b]$, $\exists\, d_x \in (x, b)$ such that $\dfrac{f(x) - f(b)}{x - b} = f'(d_x)$. Then $f'(b) = \lim_{x \to b} \dfrac{f(x) - f(b)}{x - b} = \lim_{x \to b} f'(d_x) = \lim_{d_x \to b} f'(d_x) = M$.

8. $f(x) = \sqrt{x}$ is uniformly continuous on $[0, 1]$ by Theorem 4.4. f has an unbounded derivative on $(0, 1]$ by the solution to Exercise 5.1.1.

9. (a) Suppose x_1 and x_2 are two distinct fixed points of f. By the MVT, $\exists\, c$ between x_1 and x_2 such that $x_2 - x_1 = f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$, and so $f'(c) = 1$. Since $|f'(x)| < 1\ \forall\, x \in \mathbb{R}$, this contradiction $\Rightarrow f$ has at most one fixed point.
(b) Let $f(x) = x + (1 + e^x)^{-1}$. By direct calculation, $f'(x) = 1 - \dfrac{e^x}{(1 + e^x)^2}$. Since $0 < \dfrac{e^x}{(1 + e^x)^2} < 1$, $0 < f'(x) < 1$, and so f satisfies the hypothesis of (a). If $f(c) = c$ for some $c \in \mathbb{R}$, then $\dfrac{1}{1 + e^c} = 0$, a contradiction. Therefore, f has no fixed point.

10. By Example 5.3, $x > \sin x\ \forall\, x > 0$. Since sine is an odd function, $-x > \sin(-x) = -\sin x\ \forall\, x < 0$. So $x < \sin x\ \forall\, x < 0$.

11. Let $x > 1$ and let $f(t) = \ln t$ on $[1, x]$. By the MVT, $\exists\, c \in (1, x)$ such that $f(x) = f(x) - f(1) = f'(c)(x - 1)$ or equivalently, $\ln x = \dfrac{1}{c}(x - 1)$. Since $1 < c < x$, $\dfrac{1}{x} < \dfrac{1}{c} < 1$, and so $\dfrac{x - 1}{x} < \dfrac{x - 1}{c} = \ln x < x - 1$. If $x > 0$, then $1 + x > 1$. From the last inequality, $\ln(1 + x) < (1 + x) - 1 = x$.

12. Since $\frac{y}{x} > 1$, Exercise 11 implies that

$$\frac{\frac{y}{x} - 1}{\frac{y}{x}} < \ln \frac{y}{x} < \frac{y}{x} - 1$$

or $1 - \frac{x}{y} < \ln y - \ln x < \frac{y}{x} - 1$.

Alternatively, by the MVT, $\ln \frac{y}{x} = \ln \frac{y}{x} - \ln 1 = \frac{1}{c}\left(\frac{y}{x} - 1\right)$ where $1 < c < \frac{y}{x}$. Since $\frac{x}{y} < \frac{1}{c} < 1$, the result follows.

13. By the MVT, $\cos\left(\frac{\pi}{2}\right) - \cos x = (-\sin c)\left(\frac{\pi}{2} - x\right)$ where $x < c < \frac{\pi}{2}$. Hence $\cos x = (\sin c)\left(\frac{\pi}{2} - x\right)$. Since $\sin c < 1$, $\cos x < \frac{\pi}{2} - x$ or $x + \cos x < \frac{\pi}{2}$. Since $\sin x < \sin c$, $\cos x > (\sin x)\left(\frac{\pi}{2} - x\right)$, and so $\cot x = \frac{\cos x}{\sin x} > \frac{\pi}{2} - x$. Therefore, $x + \cot x > \frac{\pi}{2}$.

Some instructors may wish to assign the following as an exercise. For $x, y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ show that $|\tan x + \tan y| \geq |x + y|$. First note that if $x = -y$, then both sides of the inequality are 0. Let $x \neq -y$. By the MVT $\exists$ c between x and $-y$ such that $\tan x - \tan(-y) = \left(\sec^2 c\right)(x - (-y))$, or equivalently, $\tan x + \tan y = \left(\sec^2 c\right)(x + y)$. Note that $\sec^2 c \geq 1$. If $x + y > 0$, then $\tan x + \tan y \geq x + y$; while if $x + y < 0$, then $\tan x + \tan y \leq x + y$. In either case, $|\tan x + \tan y| \geq |x + y|$.

5.3 Taylor's Theorem

1. Successively taking derivatives gives $\sin x, \cos x, -\sin x, -\cos x, \sin x$, etc. For $f(x) = \sin x$, $f^{(n)}(0) = 0$ for $n = 0, 2, 4, 6, \ldots$ and $f'(0) = 1, f'''(0) = -1, f^{(5)}(0) = 1, f^{(7)}(0) = -1$. Hence,

$$\begin{aligned} P_7(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(7)}(0)}{7!}x^7 \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}, \end{aligned}$$

and $R_7(x) = \frac{f^{(8)}(c)x^8}{8!} = \frac{(\sin c)x^8}{8!}$ for some c between 0 and x.

2. For $f(x) = \cos x$, using the derivatives given in Exercise 1, $f^{(n)}(0) = 0$ for n odd and $f(0) = 1, f''(0) = -1, f^{(4)}(0) = 1, f^{(6)}(0) = -1$. Hence, $P_7(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$ and $R_7(x) = \frac{(\cos c)x^8}{8!}$ for some c between 0 and x.

3. For $f(x) = \ln(1+x)$, $x \geq 0$, we obtain $f'(x) = \frac{1}{1+x} = (1+x)^{-1}, f''(x) = -(1+x)^{-2}, f'''(x) = 2(1+x)^{-3}, f^{(4)}(x) = -3!\,(1+x)^{-4}, \ldots, f^{(k)}(x) = (-1)^{(k-1)}(k-1)!(1+x)^{-k}$. So $f(0) = 0, f'(0) = 1, f''(0) = -1, f'''(0) = 2, f^{(4)}(0) = -3!, f^{(5)}(0) = 4!, f^{(6)}(0) = -5!$, and $f^{(7)}(0) = 6!$. Thus,

$$\begin{aligned} P_7(x) &= 0 + x - \frac{x^2}{2!} + \frac{2x^3}{3!} - \frac{3!x^4}{4!} + \frac{4!x^5}{5!} - \frac{5!x^6}{6!} + \frac{6!x^7}{7!} \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7}, \end{aligned}$$

and $R_7(x) = \frac{-7!(1+c)^{-8}}{8!}x^8 = \frac{-x^8}{8(1+c)^8}$ for some c between 0 and x.

4. Let $f(x) = \sin x$ or $f(x) = \cos x$. Then $f^{(n+1)}(c)$ is either $\pm \sin c$ or $\pm \cos c$, and so $\left|f^{(n+1)}(c)\right| \leq 1$. Therefore,

$$|R_n(x)| = \left|\frac{f^{(n+1)}(c)x^{n+1}}{(n+1)!}\right| \leq \frac{|x|^{n+1}}{(n+1)!} \leq \frac{1}{(n+1)!}$$

for $|x| \leq 1$. We want n so that $\frac{1}{(n+1)!} < 10^{-6}$ or equivalently, $(n+1)! > 10^6$. Therefore, $n = 9$ will do.

Let $f(x) = \ln(1+x)$, $0 \leq x \leq 1$. Since $f^{(n+1)}(c) = \frac{(-1)^n n!}{(1+c)^{n+1}}$,

$$|R_n(x)| = \left|\frac{n!}{(1+c)^{n+1}}\frac{x^{n+1}}{(n+1)!}\right| = \frac{|x|^{n+1}}{(n+1)(1+c)^{n+1}} \leq \frac{1}{(n+1)(1+c)^{n+1}}$$

for $0 \leq x \leq 1$ and $0 \leq c \leq x$. Since $1 + c \geq 1$, $\frac{1}{1+c} \leq 1$ and so $|R_n(x)| \leq \frac{1}{n+1}$. We want n so that $\frac{1}{n+1} < 10^{-6}$ or equivalently, $n + 1 > 10^6$. Therefore, $n = 10^6$ will do.

5. Choose $r \in \mathbb{R}$ with $L < r < 1$. Since $\frac{a_{n+1}}{a_n} \to L$, $\exists\ n_0 \in \mathbb{N}$ such that $n \geq n_0 \Rightarrow \frac{a_{n+1}}{a_n} < r$ or equivalently, $n \geq n_0 \Rightarrow a_{n+1} < ra_n$. Therefore $a_{n_0+1} < ra_{n_0}, a_{n_0+2} < ra_{n_0+1} < r^2 a_{n_0}, a_{n_0+3} < ra_{n_0+2} < r^3 a_{n_0}, \ldots, 0 < a_{n_0+k} < r^k a_{n_0}\ \forall\ k \in \mathbb{N}$. Since $0 < r < 1$, $r^k \to 0$. By the Squeeze Theorem, $a_{n_0+k} \to 0$, and so $a_n \to 0$. (This is part of the Ratio test of infinite series. From Chapter 7, if $\sum a_n$ converges, then $a_n \to 0$.)

6. (a) For $f(x) = e^x$, $|R_n(x)| = \frac{e^c|x|^{n+1}}{(n+1)!}$ (see Example 5.6) for some c between 0 and x. Let $x \neq 0$ and let $a_n = \frac{|x|^{n+1}}{(n+1)!}$. Then $\frac{a_{n+1}}{a_n} = \frac{|x|^{n+2}}{(n+2)!} \cdot \frac{(n+1)!}{|x^{n+1}|} = \frac{|x|}{n+2} \to 0 < 1$, and so $a_n \to 0$ by Exercise 5. Since e^c is a constant, $|R_n(x)| \to 0$ and so $R_n(x) \to 0$.

   For $f(x) = \sin x$ or $f(x) = \cos x$, $|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}\ \forall\ x \in \mathbb{R}$ by Exercise 4. As above, $R_n(x) \to 0$.

   (b) For $f(x) = \ln(1+x), 0 \leq x \leq 1$, by Exercise 4, $|R_n(x)| = \frac{|x|^{n+1}}{(n+1)(1+c)^{n+1}} \leq \frac{|x|^{n+1}}{n+1} \leq \frac{1}{n+1} \to 0$. Therefore, $R_n(x) \to 0$ for $0 \leq x \leq 1$.

7. Let $f(x) = e^x$ with $x > 0$. By Taylor's Theorem with $n = 1$, $e^x = 1 + x + \frac{e^c x^2}{2}$ where $0 < c < x$. Since $1 = e^0 < e^c < e^x$, $1 + x + \frac{x^2}{2} < e^x < 1 + x + \frac{x^2}{2}e^x$.

8. Following the directions given in this exercise, we have

$$f(x) = f(x_0) + f'(x_0)(x - x_0) - \int_{x_0}^{x} f''(t)(t - x)dt.$$

Integrating by parts with

$$\begin{array}{ll} u = f''(t) & du = f'''(t)dt \\ dv = (t-x)dt & v = \dfrac{(t-x)^2}{2} \end{array},$$

we obtain

$$\begin{aligned}
f(x) &= f(x_0) + f'(x_0)(x-x_0) - \left[\frac{f''(t)(t-x)^2}{2}\Big|_{x_0}^{x} - \int_{x_0}^{x} \frac{f'''(t)(t-x)^2}{2} dt \right] \\
&= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x_0-x)^2 + \int_{x_0}^{x} \frac{f'''(t)(t-x)^2}{2} dt.
\end{aligned}$$

Letting

$$\begin{array}{ll} u = f'''(t) & du = f^{(4)}(t)dt \\ dv = \dfrac{(t-x)^2}{2}dt & v = \dfrac{(t-x)^3}{2\cdot 3} \end{array} ,$$

$$\begin{aligned}
f(x) &= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 \\
&\quad + \frac{f'''(t)(t-x)^3}{3!}\Big|_{x_0}^{x} - \int_{x_0}^{x} \frac{f^{(4)}(t)(t-x)^3}{3!} dt \\
&= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 \\
&\quad - \frac{f'''(x_0)}{3!}(x_0-x)^3 - \int_{x_0}^{x} \frac{f^{(4)}(t)(t-x)^3}{3!} dt.
\end{aligned}$$

Letting

$$\begin{array}{ll} u = f^{(4)}(t) & du = f^{(5)}(t)dt \\ dv = \dfrac{(t-x)^3}{3!}dt & v = \dfrac{(t-x)^4}{3!\cdot 4} \end{array} ,$$

$$\begin{aligned}
f(x) &= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 \\
&\quad - \left[\frac{f^{(4)}(t)(t-x)^4}{4!}\Big|_{x_0}^{x} - \int_{x_0}^{x} \frac{f^{(5)}(t)(t-x)^4}{4!} dt \right] \\
&= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 \\
&\quad + \frac{f^{(4)}(x_0)}{4!}(x-x_0)^4 + \int_{x_0}^{x} \frac{f^{(5)}(t)(t-x)^4}{4!} dt.
\end{aligned}$$

Continuing to integrate by parts, we obtain

$$\begin{aligned}
f(x) &= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 \\
&\quad + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + (-1)^n \int_{x_0}^{x} \frac{f^{(n+1)}(t)(t-x)^n}{n!} dt.
\end{aligned}$$

(The last integral exists since $f^{(n+1)}$ is continuous. From the theory in Chapter 6 together with the continuity of $f^{(n+1)}$, one can show that the two forms of the remainder are equivalent.)

5.4 L'Hôpital's Rule

1.

$$\begin{aligned}
\lim_{x\to 0+} \frac{\cot x}{\ln x} &= \lim_{x\to 0+} \frac{-\csc^2 x}{\frac{1}{x}} && \text{(by L'Hôpital)} \\
\left(\tfrac{\infty}{-\infty} \text{ form}\right) & \\
&= -\lim_{x\to 0+} \frac{x}{\sin^2 x} && (\tfrac{0}{0} \text{ form}) \\
&= -\lim_{x\to 0+} \frac{1}{2\sin x \cos x} && \text{(by L'Hôpital)} \\
&= -\infty.
\end{aligned}$$

2.

$$\begin{aligned}
\lim_{x\to 0^+} x^2 \ln x &= \lim_{x\to 0+} \frac{\ln x}{\frac{1}{x^2}} && (\tfrac{-\infty}{\infty} \text{ form}) \\
(0\cdot -\infty \text{ form}) & \\
&= \lim_{x\to 0+} \frac{\frac{1}{x}}{\frac{-2}{x^3}} && \text{(by L'Hôpital)} \\
&= -\lim_{x\to 0+} \frac{x^2}{2} \\
&= 0.
\end{aligned}$$

3. $\lim\limits_{x\to 0^+} x^x = \lim\limits_{x\to 0^+} e^{x\ln x} = e^{\lim_{x\to 0^+} x \ln x}$. Since
(0^0 form)

$$\begin{aligned}
\lim_{x\to 0^+} x \ln x &= \lim_{x\to 0+} \frac{\ln x}{\frac{1}{x}} && (\tfrac{-\infty}{\infty} \text{ form}) \\
(0\cdot -\infty \text{ form}) &
\end{aligned}$$

$$\begin{aligned} &= \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} \quad \text{(by L'Hôpital)} \\ &= \lim_{x\to 0^+} (-x) \\ &= 0, \end{aligned}$$

$\lim_{x\to 0^+} x^x = e^0 = 1.$

4. $\lim_{x\to 0^+} (\cos\sqrt{x})^{\frac{1}{x}} = \lim_{x\to 0^+} e^{\frac{1}{x}\ln(\cos\sqrt{x})} = e^{\lim_{x\to 0^+}\frac{1}{x}\ln(\cos\sqrt{x})}$. Since
(1^∞ form)

$$\begin{aligned} \lim_{x\to 0^+} \frac{\ln(\cos\sqrt{x})}{x} &= \lim_{x\to 0+} \frac{(-\sin\sqrt{x})\left(\frac{1}{2\sqrt{x}}\right)}{\cos\sqrt{x}} \quad \text{(by L'Hôpital)} \\ &= -\frac{1}{2}\lim_{x\to 0^+}\frac{\tan\sqrt{x}}{\sqrt{x}} \quad \left(\frac{0}{0}\text{ form}\right) \\ &= -\frac{1}{2}\lim_{x\to 0^+}\frac{(\sec^2\sqrt{x})\frac{d}{dx}\sqrt{x}}{\frac{d}{dx}\sqrt{x}} \quad \text{(by L'Hôpital)} \\ &= -\frac{1}{2}\lim_{x\to 0^+}\sec^2\sqrt{x} \\ &= -\frac{1}{2}(1) \\ &= -\frac{1}{2}, \end{aligned}$$

$\lim_{x\to 0^+} (\cos\sqrt{x})^{\frac{1}{x}} = e^{-\frac{1}{2}}.$

5.

$$\begin{aligned} \underset{\left(\frac{0}{0}\text{ form}\right)}{\lim_{x\to 0}\frac{x^2}{1-\cos x}} &= \underset{\left(\frac{0}{0}\text{ form}\right)}{\lim_{x\to 0}\frac{2x}{\sin x}} \quad \text{(by L'Hôpital)} \\ &= \lim_{x\to 0}\frac{2}{\cos x} \quad \text{(by L'Hôpital)} \\ &= 2. \end{aligned}$$

6. $\lim_{x\to\infty} \frac{e^x}{\pi^x} = \lim_{x\to\infty} \left(\frac{e}{\pi}\right)^x = 0$ since $0 < \frac{e}{\pi} < 1$.

7.

$$\begin{aligned}
\underset{(\infty-\infty \text{ form})}{\lim_{x\to\frac{\pi}{2}^-} (\sec x - \tan x)} &= \lim_{x\to\frac{\pi}{2}^-} \frac{1-\sin x}{\cos x} && \left(\frac{0}{0}\text{ form}\right) \\
&= \lim_{x\to\frac{\pi}{2}^-} \frac{-\cos x}{-\sin x} && \text{(by L'Hôpital)} \\
&= \frac{0}{1} \\
&= 0.
\end{aligned}$$

8. $\underset{(1^\infty \text{ form})}{\lim_{x\to\infty} \left(1+\frac{3}{x}\right)^x} = \lim_{x\to\infty} e^{x\ln\left(1+\frac{3}{x}\right)} = e^{\lim_{x\to\infty} x\ln\left(1+\frac{3}{x}\right)}$. Since

$$\begin{aligned}
\underset{(\infty\cdot 0 \text{ form})}{\lim_{x\to\infty} x\ln\left(1+\frac{3}{x}\right)} &= \lim_{x\to\infty} \frac{\ln\left(1+\frac{3}{x}\right)}{\frac{1}{x}} && \left(\frac{0}{0}\text{ form}\right) \\
&= \lim_{x\to\infty} \frac{\frac{-\frac{3}{x^2}}{1+\frac{3}{x}}}{-\frac{1}{x^2}} && \text{(by L'Hôpital)} \\
&= \lim_{x\to\infty} \frac{3}{1+\frac{3}{x}} \\
&= 3,
\end{aligned}$$

$$\lim_{x\to\infty} \left(1+\frac{3}{x}\right)^x = e^3.$$